Methods of Thermal Stress Calculation for Circular Cylinders and Disks
— Comparison of Results by Timoshenko's and Poritsky's Methods and the Equivalence of the Methods —

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Two methods of thermal stress calculation by Timoshenko's method and Poritsky's method for circular cylinders and disks were discussed, and the results of calculation were compared. The first method was given by Timoshenko based on the integration of a temperature × distance from central axis function, and the other was that given by Poritsky based on multi-variant simultaneous equations which represent elastical balances of principal stresses and strains in a cylinder and a disk. Calculations were carried out for the following four cases: a solid cylinder, a cylinder with a concentric hole, a solid disk and a disk with a concentric hole. The results obtained by the two methods were identical for the four cases. It was proved that the two methods were mathematically equivalent.

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1. Introduction

The following methods were used to calculate stresses in composites caused by difference in thermal properties of component materials:

(a) Equations[1a-1d] for thermal stress calculation were applied by replacing thermal expansion difference caused by temperature difference with that caused by the difference in expansion coefficient.[1e, 2-4]

(b) Equations[5-9] to calculate stresses in composites formed by fusion or adhesion are adopted.

As stated above, there is a close relationship between these two methods so that method b) can also be applied to the calculation of thermal stress.[10] When this was performed with a cylinder and disk, it was found that the results agree very well with those of method a). Consequently, these methods were further examined in detail to prove that they are mathematically equivalent. The present paper describes this process.

2. Calculation of Thermal Stress of Cylinder and Disk: Timosheko's Method

2.1. Case of Cylinder[1a]

Thermal stresses generated in a cylinder with axisymmetric temperature distribution are represented by Eqs.(1.1)-(1.3) by Timoshenko et al.

Radial stress:

$$\sigma_r = \frac{\alpha E}{1-\nu}\left(\frac{1}{b^2}K - \frac{1}{r^2}\int_0^r T\cdot r dr\right) \quad \cdots\cdots\cdots (1.1)$$

Tangential stress:

$$\sigma_\theta = \frac{\alpha E}{1-\nu}\left(\frac{1}{b^2}K + \frac{1}{r^2}\int_0^r T\cdot r dr\right) \quad \cdots\cdots\cdots (1.2)$$

Axial stress:

$$\sigma_z = \frac{\alpha E}{1-\nu}(\text{constant} - T)$$

When the constant is determined with the assumption that the integration of axial stress across the cross section is zero[10] . . . . Condition (T1);

the following result is obtained:

Axial stress:

$$\sigma_z = \frac{\alpha E}{1-\nu}\left(\frac{2}{b^2}K - T\right) \quad \cdots\cdots\cdots\cdots (1.3)$$

in these cases

$$K = \int_0^b T\cdot r dr$$

where b is the radius of the cylinder, r is a distance from the center; T is temperature as a function of r, σ is stress; the subscripts of σ denote r: radial direction; θ: tangential direction, and z: axial direction; α is the coefficient of thermal expansion, E is Young's modulus; and ν is Poisson's ration. A difference in T with location causes that in thermal expansion $\alpha\cdot T$, causing thermal stress.

2.2. Case of Disk[1b]

When introducing the equations in the previous section, if the axial stress is assumed to be zero, the equations to represent the stresses in the disk are generated:

$$\sigma_r = \alpha E\left(\frac{1}{b^2}K - \frac{1}{r^2}\int_0^r T\cdot r dr\right)$$

$$\sigma_\theta = \alpha E\left(\frac{1}{b^2}K + \frac{1}{r^2}\int_0^r T\cdot r dr - T\right)$$

2.3. Cases with Holes

Assuming that σ_r is zero on the inner surfaces of holes, equations can also be derived for a cylinder and disk with concentric holes.

2.4. Calculation With Computer

If $T\cdot r$ is an integrable function with respect to r on a computer language, both programming and calculation are easy. Even if the temperature distribution is represented by continuously joined lines, or is represented by lines which are connected stepwise, $T{\cdot}r$ is integrable and stress calculation can be performed.

2.5. Application of Calculation

These equations can be used to calculate stress in a heterogeneous material when αT is replaced with a difference in the expansion characteristic depending on location,[1e,3] and that in thermally tempered glass plate when it is replaced with a difference in expansion caused by nonuniform thermal hysteresis.[10]

3. Calculation of Thermal Stress of Cylinder and Disk: Poritsky's Method[5-9]

3.1. Case of Solid Cylinder

Assuming that each layer in a cylindrical seal is equivalent with a tube or a cylinder subjected to pressures at inner, outer and end surfaces, respectively (Fig.1), the stress distribution in the seal is expressed in the following forms:

$$\sigma_r = A - B/r^2 \quad \ldots\ldots (2.1)$$

$$\sigma_\theta = A + B/r^2 \quad \ldots\ldots (2.2)$$

$$\sigma_z = C \quad \ldots\ldots (2.3)$$

where A, B, and C are constants allocated for each layer.

Using these equations, the following conditions are written:

Conditions for the balance of force inside the seal:

stress is finite at $r = 0$
. . . . Condition (P1);

σ_r is continuous at the boundaries
. . . . Condition (P2);

$\sigma_r = 0$ at the outer surface
. . . . Condition (P3);

the integration of σ_z across the cross section is zero
. . . . Condition (P4).

Conditions of no fracture st boundaries and in layers require continuities of strains and displacements. Continuity of tangential strain e_θ at boundaries gives:

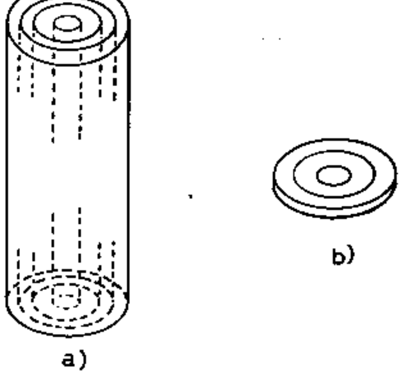

Fig. 1. Multi-layered a) cylindrical hermetic seal and b) disk-like seal.

$$e_{\theta 1} \equiv \frac{1}{E}\{\sigma_{\theta 1} - \nu(\sigma_{r1} + \sigma_{z1})\} + \alpha T_1$$
$$= \frac{1}{E}\{\sigma_{\theta 2} - \nu(\sigma_{r2} + \sigma_{z2})\} + \alpha T_2 \equiv e_{\theta 2}$$

. . . . Condition (P5)

continuity of axial strain e_z:

$$e_{z1} \equiv \frac{1}{E}\{\sigma_{z1} - \nu(\sigma_{r1} + \sigma_{\theta 1})\} + \alpha T_1$$
$$= \frac{1}{E}\{\sigma_{z2} - \nu(\sigma_{r2} + \sigma_{\theta 2})\} + \alpha T_2 \equiv e_{z2}$$

. . . . Condition (P6)

Using these conditions, multi-variant, simultaneous linear equations with A, B, and C as unknowns are established; and stress in the cylinder can be calculated by solving them and determining A, B, and C. Here, subscripts 1 and 2 designate two adjacent layers.

In this derivation, thermal stress can be obtained when a difference in expansion by a difference in materials is replaced with that by a difference in temperature.

3.2. Case of Disk[10]

When all C's are set to zero in the process of establishing equations in the previous section, a solution for a disk can be obtained.

3.3. Cases with Concentric Hole

Equation can be derived using the same assumption as in **Section 2.3.**

4. Examples of Calculation

First, the examples of calculation were described.

The material properties were assumed as follows: $E = 7000\text{kg/mm}^2$, $\alpha = 100 \times 10^{-7}{}^\circ\text{C}^{-1}$, and $\nu = 0.25$. These values were close to those of soda lime glass. For the convenience of comparison, the maximum temperature difference was set to 100°C. The stress values in this case were within 7kg/mm^2.

4.1. Points in Calculation

In Poritsky's method, temperature distribution is limited in which temperature changes stepwise and discontinuously. Consequently, the same stepwise temperature distribution was also adopted in Timoshenko's method, although it could accept other types of temperature distribution, if it was univalent, finite and $T{\cdot}r$ was integrable.

In applying Poritsky's method to thermal stress calculation, the same values are given as the properties of all layers, so that the "division by zero" error often appears. To counter this, there are the following measures: the order of arranging equations is changed in the manner of trial and error; the order of arranging variables is changed in the manner of trial and error; and material constants are allowed to fluctuate unless they do not affect conclusions (within 2% in this study). Here, the last method was adopted with respect to E.

4.2. Verification by Calculation Examples

In the case of thermal stress, an (r/b)-T relation determines an (r/b)-(stress) relation. Therefore, in **Figs.2-5**, not the absolute value of r but only the central axis ($r = 0$) and peripheral ($r = b$) were shown for the x-axis, while stress was given for

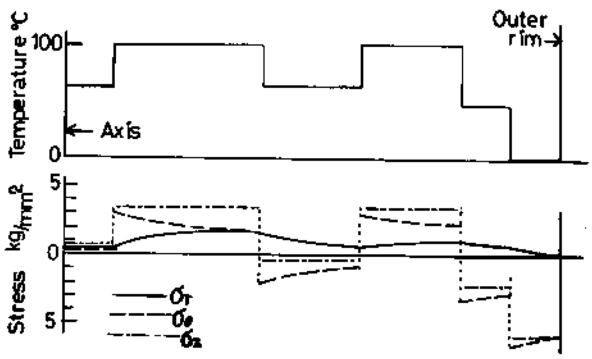

Fig. 2. Thermally induced radial stress σ_r, tangential stress σ_θ and axial stress σ_z distributions along radius in a solid cylinder. The temperature distribution used for the calculation is shown in the upper part of the figure.

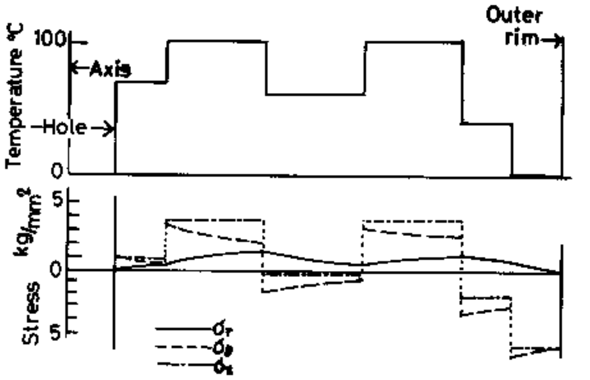

Fig. 3. Thermally induced radial stress σ_r, tangential stress σ_θ and axial stress σ_z distributions along radius in a cylinder with a concentric hole.
The temperature distribution used for the calculation is shown in the upper part of the figure.

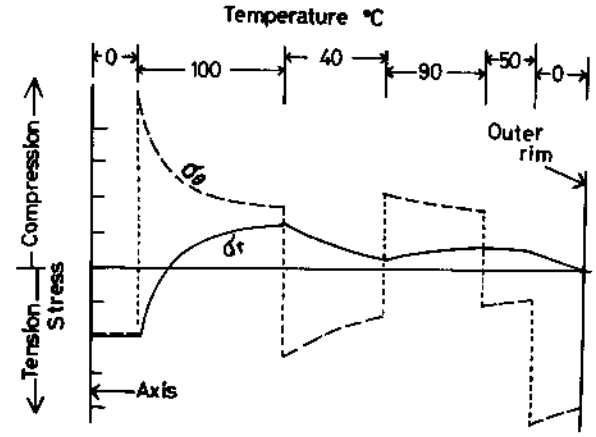

Fig. 4. Thermally induced radial stress σ_r and tangential stress σ_θ distributions along radius in a solid disk. The temperature distribution used for the calculation is shown in the upper part of the figure. Stress : 1kg/mm^2/division.

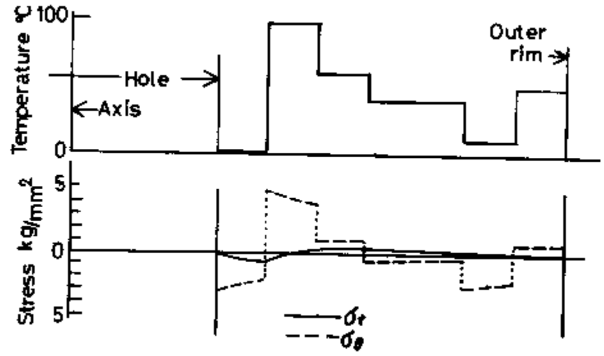

Fig. 5. Thermally induced radial stress σ_r and tangential stress σ_θ distributions along radius in a disk with a concentric hole. The temperature distribution used for the calculation is shown in the upper part of the figure.

the y-axis. An example of a solid cylinder is presented in Fig.2, that of a cylinder with a concentric hole in Fig.3, that of a solid disk in Fig.4, and that of a disk with a concentric hole in Fig.5. In each case, calculation results by the two methods agreed well, so the methods of calculation are not distinguished in the figure.

5. Verification of Mathematical Equivalence

In the first step, it is verified that the two methods are equivalent in the case of stepwise temperature distribution. With respect to other temperature distributions, it can also be said that both are equivalent, considering that they are the limit of fine graduation of the temperature axis of the stepwise distribution.

5.1. Comparison of Equation Forms

First, it should be noted that the integration of $T{\cdot}r$ over the interval of $0{\sim}r$ is a continuous function. In **Fig.6** where a part of the stepwise temperature distribution curve is provided, where the temperature is constant (T_s) in the interval of G~H ($r = g{\sim}r = h$) on the radial axis, and both neighbors have different temperature values. Assuming that there is a general point J (The value of a radius is set to a general value r) in the interval of G~H, stress on the point J is discussed. The first terms in the parentheses of Eqs.(1.1) and (1.2) are constant K.

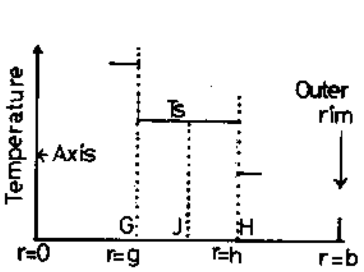

Fig. 6. A schematic representation of a part of a step-like temperature distribution curve.

When the integration of the second term is rewritten by separation into integration in the range (r = 0~point G) and that in the interval (point G~ point J), the following equation is obtained:

$$\frac{1}{r^2}\int_0^r T{\cdot}r dr = \frac{1}{r^2}\int_0^g T{\cdot}r dr + \frac{1}{r^2}\int_g^r T_s{\cdot}r dr$$

$$= \frac{Q}{r^2} + \frac{1}{r^2}\left[\frac{T_s}{2}r^2\right]_g^r$$

$$= \frac{1}{r^2}\left(Q - \frac{T_s}{2}g^2\right) + \frac{T_s}{2}$$

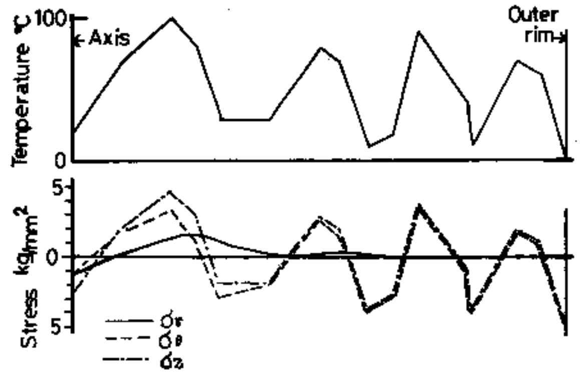

Fig. 7. A test calculation of thermal stresses in a solid cylinder with a continuous temperature distribution.

where $Q = \int_0^g T \cdot r dr$ is a constant in the interval of (G~H). This result is utilized to introduce the following relations:

$$\sigma_r = A - B/r^2,$$
$$\sigma_\theta = A + B/r_2$$

with

$$A = \frac{\alpha E}{1-\nu}\left(\frac{1}{b^2}K - \frac{T_s}{2}\right)$$
$$B = \frac{\alpha E}{1-\nu}\left(Q - \frac{T_s}{2}g^2\right),$$

which are identical in formula with Eqs.(2.1) and (2.2). The Eq.(1.3) is a constant in the interval (G~H), and this constant C is given by:

$$C = \frac{\alpha E}{1-\nu}\left(\frac{2}{b^2}K - T_s\right)$$

and therefore, σ_z is expressed by Eq.(2.3).

5.2. Agreement of Boundary and Balance Conditions

Equation (1.1) is a continuous function and satisfies condition (P1). In addition, it is obvious that it satisfies conditions (P2) and (P3). When Eqs.(1.1) and (1.3) are substituted into conditional Eqs.(P5) and (P6) for testing, it is found that the Eqs.(1.1)-(1.3) satisfy conditional Eqs.(P5) and (P6). Finally, condition (T1) is the same as the condition (P4).

5.3. Equivalence

As stated above, in the case of stepwise temperature distribution, stresses by Timoshenko's method are expressed by the same equations in the form as those of Poritsky's method, and they satisfy all of the conditional equations required by Poritsky's method. Therefore, stresses by Timoshenko's method are that obtained as solutions by Poritsky's method at the same time. In other words, both methods are mathematically equivalent and equal. This is also true even with a large number of intervals of temperature distribution and the narrow width of each interval.

The temperature distribution in a general form can be regarded as the limit of stepwise distribution with narrow intervals so that the same conclusion can also be obtained in this case.

Figure 7 presents an example of calculating the thermal stresses of a cylinder by Timoshenko's method with respect to a continuous temperature distribution without steps.

Also, in the case of a disk, the same conclusion can be derived by a similar process.

Furthermore, the same process and conclusion can be applied to a cylinder and disk with concentric holes.

6. Conclusion

It was proved that the expressions of stresses and the processes of calculation in Timoshenko's method and Poritsky's method are apparently quite different, and yet detailed study reveals that both are mathematically equivalent. It can be said that this information has further clarified the essence of the two methods. As stated in the Introduction, these methods have an extensive application range, and further utilization is recommended.

With some modification, the methods are also applicable to solid and hollow spheres.

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